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# MULTISUMMABILITY OF FORMAL POWER SERIES SOLUTIONS OF NONLINEAR MEROMORPHIC DIFFERENTIAL EQUATIONS

by Boele L. J. BRAAKSMA

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## 0. Introduction.

In this paper we consider nonlinear ordinary differential equations

$$(0.1) \quad x^{v+1} \frac{dy}{dx} = F(x, y)$$

where  $x \in \mathbb{C}$ ,  $y \in \mathbb{C}^n$ ,  $v \in \mathbb{N}$ ,  $v > 0$  and  $F$  is an analytic function in a neighborhood of  $(0, a) \in \mathbb{C} \times \mathbb{C}^n$ . If (0.1) has a formal power series solution

$$(0.2) \quad \hat{y}(x) = \sum_{m=0}^{\infty} c_m x^m, \quad c_0 = a$$

then we will show that  $\hat{y}$  can be summed by a new injective summation procedure, called multisummability, introduced by Ecalle (cf. [7], [8]). We use the description of this procedure given by Martinet and Ramis [13]. Equivalent forms of multisummability have been given by Balser [1], [2], Jurkat [10], Malgrange and Ramis [12], [15].

This multisummability property of formal power series solutions has been announced by Ecalle during the « Journées Résurgentes » in Paris 1989, with a rough idea of a proof. Later Ramis [14] announced several conjectures of which the statement concerning (0.1) given above

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constitutes the main conjecture. He gave a sketch of the principal ideas of a complete proof using essentially the techniques and ideas of his paper with Sibuya [16] extended to analytic infinitesimal neighborhoods of the origin. Here we give a proof in the spirit of Ecalle's work (cf. [6]) using convolution equations similar to our treatment of the linear case in [5] and to the nonlinear case with one level only in [4]. A different treatment of the linear case occurs in [3] and [13].

The organization of this paper is as follows. In section 1 we give a concise review of the definition of multisummability. In section 2 we reduce equation (0.1) to a normal form  $D[y] = 0$  (cf. (2.2)) from which the different levels  $k_1 > k_2 > \dots > k_r > 0$  of this equation can be read off. These levels are closely associated with the Newton polygon of (0.1). To each level correspond singular values and singular directions (cf. definition in section 2). The formal power series solutions  $\hat{y}$  of  $D[y] = 0$  is  $(k_1, \dots, k_r)$ -summable in nonsingular directions to an analytic solution. In theorem 1 in section 2 this result is formulated for a somewhat more general case where  $F$  (cf. (0.1)) itself is the sum of a  $(k_1, \dots, k_r)$ -summable power series  $\hat{F}(x, y)$  in  $x$  with coefficients that are analytic in  $y$ .

In section 3 we derive convolution equations  $Q_j \psi = 0$  which arise by application of some form of Borel transformation of order  $k_j$  to  $D[y] = 0$ ,  $j = 1, \dots, r$ . Here  $Q_j$  and  $Q_{j-1}$  are connected by an acceleration operator (cf. (7.2)). An analysis of these convolution equations leads to a proof of the main result which is given in section 3. This proof consists of four steps which are formulated as lemmas in that section: First it is shown that the convolution equation  $Q_r \psi = 0$  corresponding to the lowest level has an analytic solution  $\psi_r$  in a neighborhood of the origin (lemma 2) and we show that  $\psi_r$  can be analytically continued on a sector (lemma 3) and has a certain exponential growth order  $\mu_r$  (lemma 4). Since  $\mu_r > k_r$  if  $r > 1$  the Laplace transform of order  $k_r$  of  $\psi_r$  does not exist in general (it would have led to a Borel sum  $y$  of  $\hat{y}$ ), but the growth order  $\mu_r$  is such that we may apply the acceleration operator from level  $k_r$  to level  $k_{r-1}$  to  $\psi_r$  (lemma 5). Thus we obtain a solution  $\psi_{r-1}$  of  $Q_{r-1} \psi = 0$ . Repeating this procedure with lemmas 2-5 we obtain finally a solution  $\psi_1$  of  $Q_1 \psi = 0$ , the convolution equation of highest level  $k_1$  which appears to have exponential growth of order  $\leq k_1$ . Therefore its Laplace transform  $y$  of order  $k_1$  exists and this function  $y$  is analytic solution of  $D[y] = 0$ , it is the  $(k_1, \dots, k_r)$ -sum of  $\hat{y}$ .

Lemma 2 is proven in section 4 by means of the contraction principle. Lemma 3 concerning the analytic continuation of solutions is derived in section 5 from a linearization of the convolution equations. In section 6 we prove the exponential estimates of the solutions  $\psi_j$  (Lemma 4) by means of a majorant equation. Section 7 contains the proof of lemma 5 concerning the acceleration of  $Q_j$  to  $Q_{j-1}$ . In section 8 we compare multisum-solutions on different sides of singular directions in view of the Stokes phenomenon.

### 1. Definition of multisummability.

In this section we give a concise review of the definitions of Laplace and Borel transforms, accelerations and multisummability as given by Martinet and Ramis [13] (cf. also Malgrange and Ramis [12], [15]).

If  $\rho > 0$ ,  $n \in \mathbb{N}$  then  $\Delta_n(0, \rho) := \{\xi \in \mathbb{C}^n \mid |\xi| < \rho\}$ . A direction  $d$  will be a half line  $\{\xi \in \mathbb{C} \mid \xi \neq 0, \arg \xi = \theta\}$  where  $\theta$  is some real number and  $\arg d := \theta$ . If  $\alpha > 0$  then  $S(d, \alpha) := \left\{ \xi \in \mathbb{C} \mid \xi \neq 0, |\arg \xi - \arg d| < \frac{1}{2}\alpha \right\}$ . A neighborhood of 0 in  $S(d, \alpha)$  will be a set  $\{\xi \in S(d, \alpha) \mid 0 < |\xi| < \rho(\arg \xi)\}$  where  $\rho$  is some positive-valued continuous function on  $\left(\arg d - \frac{1}{2}\alpha, \arg d + \frac{1}{2}\alpha\right)$ .

Let  $\mu > 0$ . We say that a function  $f$  defined on a neighborhood of  $\infty$  in  $S(d, \alpha)$  is of exponential growth of order  $\leq \mu$  if to every closed subsector  $S'$  of  $S(d, \alpha)$  corresponds a positive constant  $c$  such that  $f(\xi) = O(1) \exp |c\xi^\mu|$  as  $\xi \rightarrow \infty$  on  $S'$ .

Let  $k > 0$  and  $f: S(d, \alpha) \rightarrow \mathbb{C}$  be analytic and of exponential growth of order  $\leq k$  whereas  $f(\xi) = O(\xi^{\varepsilon-k})$  as  $\xi \rightarrow 0$  on  $S(d, \alpha)$  for some  $\varepsilon > 0$ . Then the Laplace transform of order  $k$  in the direction  $d$  of  $f$  is defined by

$$(1.1) \quad (\mathcal{L}_{k,d}f)(x) = \int_d f(\xi) \exp - (\xi/x)^k d(\xi^k),$$

where the path of integration  $d$  runs from 0 to  $\infty$ . Then  $\mathcal{L}_{k,d}f$  is analytic in a neighborhood of 0 in  $S(d, \alpha + \pi/k)$ .

Let

$$(1.2) \quad \hat{f}(\xi) = \sum_{m=1}^{\infty} c_m \xi^{m-k} \in \xi^{1-k} \mathbb{C}[[\xi]]$$

be a formal power series. Then we define

$$(1.3) \quad (\hat{\mathcal{L}}_k \hat{f})(x) := \sum_{m=1}^{\infty} c_m \Gamma(m/k) x^m \in x\mathbb{C}[[x]].$$

If  $f(\xi) \sim \hat{f}(\xi)$  as  $\xi \rightarrow 0$  in  $S(d, \alpha)$  and the assumptions above on  $f$  are satisfied then  $(\mathcal{L}_{k,d} f)(x) \sim (\hat{\mathcal{L}}_k \hat{f})(x)$  as  $x \rightarrow 0$  on  $S(d, \alpha + \pi/k)$ .

Let  $U$  be a neighborhood of 0 in  $S(d, \alpha + \pi/k)$  and  $g: U \rightarrow \mathbb{C}$  be analytic and  $g(\xi) = O(\xi^\delta)$  as  $\xi \rightarrow 0$  where  $\alpha > 0$ ,  $\delta \in \mathbb{R}$ ,  $d_\pm$  are directions in  $S(d, \alpha + \pi/k)$ ,  $\arg d_+ > \arg d + \pi/(2k)$ ,  $\arg d_- < \arg d - \pi/(2k)$ . Then the Borel transform of order  $k$  in the direction  $d$  of  $g$  is defined by

$$(1.4) \quad (\mathcal{B}_{k,d} g)(\xi) = \frac{1}{2\pi i} \int_{\gamma} g(x) \exp(\xi/x)^k d(x^{-k}),$$

where  $\gamma$  is a loop from 0 to 0 in  $U$  with the first part in direction  $d_+$  and the last part in direction  $d_-$ ,  $\xi \in S(d, \alpha)$ . Then  $\mathcal{B}_{k,d} g$  is analytic and of exponential growth of order  $\leq k$  in  $S(d, \alpha)$ . If

$$(1.5) \quad \hat{g}(x) := \sum_{m=1}^{\infty} a_m x^m \in \mathbb{C}[[x]]$$

then

$$(1.6) \quad (\hat{\mathcal{B}}_k \hat{g})(\xi) := \sum_{m=1}^{\infty} a_m \xi^{m-k} / \Gamma(m/k).$$

If  $g(x) \sim \hat{g}(x)$  as  $x \rightarrow 0$  on  $U$  and the assumptions above hold then  $(\mathcal{B}_{k,d} g)(\xi) \sim (\hat{\mathcal{B}}_k \hat{g})(\xi)$  as  $\xi \rightarrow 0$  on  $S(d, \alpha)$ . We have  $\mathcal{B}_{k,d} \mathcal{L}_{k,d} = \text{id}$ , and  $\mathcal{L}_{k,d} \mathcal{B}_{k,d} = \text{id}$  on the spaces of functions  $f$  and  $g$  which satisfy the assumptions above.

Let  $(\rho_k f)(\xi) = f(\xi^{1/k})$ . If  $f$  and  $\varphi$  are analytic in a neighborhood  $U$  of 0 in  $S(d, \alpha)$  and  $f(\xi), \varphi(\xi) = O(\xi^{\varepsilon-k})$  as  $\xi \rightarrow 0$  in  $U$  for some  $\varepsilon > 0$ , then the  $k$ -convolution of  $f$  and  $\varphi$  in  $U$  is defined by

$$(1.7) \quad (f *_k \varphi)(\xi) = \rho_k^{-1}((\rho_k f) * (\rho_k \varphi))(\xi), \quad \xi \in U.$$

If  $f$  and  $\varphi$  are also analytic and of exponential growth of order  $\leq k$  in  $S(d, \alpha)$  then  $\mathcal{L}_k(f *_k \varphi) = \mathcal{L}_k f \cdot \mathcal{L}_k \varphi$  on a neighborhood of 0 in  $S(d, \alpha + \pi/k)$ . Similarly if  $g$  and  $\psi$  satisfy the assumptions in the definition of  $\mathcal{B}_{k,d}$  on  $S(d, \alpha + \pi/k)$  then  $\mathcal{B}_k(g \psi) = (\mathcal{B}_k g) * (\mathcal{B}_k \psi)$  on  $S(d, \alpha)$ . Here  $\mathcal{L}_k = \mathcal{L}_{k,d}$  and  $\mathcal{B}_k = \mathcal{B}_{k,d}$ .

Let  $0 < k < k'$ . Then the operator of  $(k', k)$ -acceleration in the direction  $d - \mathbf{A}_{k', k; d} -$  is defined by  $\mathbf{A}_{k', k; d} = \mathcal{B}_{k', d} \mathcal{L}_{k, d}$ . This operator makes sense in the space of analytic functions  $f: S(d, \alpha) \rightarrow \mathbb{C}$  which are of exponential growth in  $S(d, \alpha)$  of order  $\leq k$  and which satisfy  $f(\xi) = O(\xi^{\varepsilon-k})$  as  $\xi \rightarrow 0$  in  $S(d, \alpha)$  for some  $\varepsilon > 0$ . However, Ecalle has shown that this operator may be extended to functions with the same conditions except that order  $\leq k$  is replaced by order  $\leq \mu$  where  $\mu^{-1} := k^{-1} - (k')^{-1}$  (cf. [6], [7], [13]). Then  $\mathbf{A}_{k', k; d} f$  is analytic in a neighborhood of 0 in  $S(d, \alpha + \pi/\mu)$ . Moreover if  $f(\xi) \sim \hat{f}(\xi)$  as  $\xi \rightarrow 0$  in  $S(d, \alpha)$  (cf. (1.2)) then

$$(1.8) \quad (\mathbf{A}_{k', k; d} f)(\xi) \sim \sum_1^{\infty} c_m \frac{\Gamma(m/k)}{\Gamma(m/k')} \xi^{m-k'} \text{ as } \xi \rightarrow 0 \text{ in } S(d, \alpha + \pi/\mu).$$

If  $f$  and  $g: S(d, \alpha) \rightarrow \mathbb{C}$  satisfy the assumptions of the definition of  $\mathbf{A}_{k', k; d} f$  and  $\mathbf{A}_{k', k; d} g$  then we have

$$(1.9) \quad \mathbf{A}_{k', k; d} (f *_{k'} g) = (\mathbf{A}_{k', k; d} f) *_{k'} (\mathbf{A}_{k', k; d} g).$$

DEFINITION 1. — **k-summability** of a formal power series in a multidirection  $\mathbf{d}$  or multisector  $\mathbf{S}$ .

Let  $n$  and  $r \in \mathbb{N}$ ,  $\hat{f} \in x\mathbb{C}[[x]]^n$ ,  $\mathbf{k} = (k_1, \dots, k_r)$ ,  $0 < k_r < \dots < k_1$ ,  $\mathbf{d} = (d_1, \dots, d_r)$ ,  $\mathbf{S} = (S_1, \dots, S_r)$  where  $d_j$  is a direction,  $S_j = S(d_j, \varepsilon_j + \pi/k_j)$ ,  $\varepsilon_j > 0$ ,  $j = 1, \dots, r$ . Let

$$(1.10) \quad k_0 := +\infty, \quad \mu_j := (k_j^{-1} - k_{j-1}^{-1})^{-1}, \quad j = 1, \dots, r.$$

Then  $\hat{f}$  is said to be **k-summable** in the multidirection  $\mathbf{d}$  or multisector  $\mathbf{S}$  if

a)  $S_{j-1} \subset S_j$ ,  $j = 2, \dots, r$ .

b)  $\mathcal{B}_{k_r} \hat{f}$  is convergent in  $\Delta_1(0; \rho) \setminus \{0\}$  for some  $\rho > 0$ . Let  $g_r$  be the sum of this series.

c) For  $j = r, r-1, \dots, 1$  respectively the function  $g_j$  can be continued analytically on  $S(d_j, \varepsilon_j)$  and is of exponential growth of order  $\leq \mu_j$  on  $S(d_j, \varepsilon_j)$ , and if  $j \neq 1$  we define  $g_{j-1} := \mathbf{A}_{k_{j-1}, k_j; d_j} g_j$  on a neighborhood of 0 in  $S(d_j, \varepsilon_j + \pi/\mu_j)$ .

Then the **k-sum** of  $\hat{f}$  in multidirection  $\mathbf{d}$  or multisector  $\mathbf{S}$  is defined by  $S_{\mathbf{k}, \mathbf{d}} \hat{f} := \mathcal{L}_{k_1, d_1} g_1$  (cf. [13]).

This sum is analytic in a neighborhood  $U$  of 0 in  $S_1$  and satisfies  $S_{\mathbf{k}, \mathbf{d}} \hat{f}(x) \sim \hat{f}(x)$  as  $x \rightarrow 0$  in  $U$ . The operator  $S_{\mathbf{k}, \mathbf{d}}$  is injective.

The definition above implies that  $\hat{f}$  is  $\mathbf{k}$ -summable on  $(I_1, \dots, I_r)$  in the sense of Malgrange and Ramis [12], [15] if  $I_j$  is a closed subsector of  $S_j$  with length  $\geq \pi/k_j$  and  $I_1 \subset I_2 \subset \dots \subset I_r$ . This may be shown using a decomposition of multisums as for example in Lemma 4 in [5].

If  $\hat{\phi} = a + \hat{f}$  where  $a$  is a constant and  $\hat{f}$  is as above we also say that  $\hat{\phi}$  is  $\mathbf{k}$ -summable in direction  $\mathbf{d}$  with  $S_{k,d}\hat{\phi} := a + S_{k,d}\hat{f}$ . If  $\hat{\phi}$  is  $\mathbf{k}$ -summable in all directions except finitely many then  $\hat{\phi}$  is said to be  $\mathbf{k}$ -summable. If  $\hat{\phi}$  is  $\mathbf{k}$ -summable in direction  $\mathbf{d}$  with  $d_1 = \dots = d_r = : d$  then  $\hat{\phi}$  is said to be  $\mathbf{k}$ -summable in direction  $d$ .

If no confusion arises we omit the directions  $d$  and  $\mathbf{d}$  in the operators  $\mathcal{L}_{k,d}$ ,  $\mathcal{B}_{k,d}$ ,  $\mathbf{A}_{k',k;d}$  and  $S_{k,d}$ .

## 2. Normal form and statement of the result.

First we reduce (0.1) to a normal form (cf. [16] and [17]). Let  $y(x) = P(x) + x^\mu \tilde{y}(x)$  where  $P(x) = \sum_{m=0}^{M+\mu-1} c_m x^m$  (cf. (0.2)) and  $N, \mu \in \mathbb{N}$  will be chosen later on. Substitution in (0.1) gives

$$(2.1) \quad x^{v+1} \tilde{y}' = \tilde{F}(x, \tilde{y})$$

where

$$\begin{aligned} \tilde{F}(x, \tilde{y}) &:= x^M \tilde{F}_0(x) + \tilde{A}(x) \tilde{y} + x^\mu \tilde{F}_2(x, \tilde{y}), \\ \tilde{F}_0(x) &:= -x^{-M-\mu} \{F(x, P(x)) - x^{v+1} P'(x)\}, \\ \tilde{A}(x) &:= -\mu x^v I + D_y F(x, P(x)), \\ \tilde{F}_2(x, \tilde{y}) &:= x^{-2\mu} \{F(x, P(x) + x^\mu \tilde{y}) - F(x, P(x)) - D_y F(x, P(x)) x^\mu \tilde{y}\}. \end{aligned}$$

Then  $\tilde{F}_2$  is analytic near  $(0,0)$  in  $\mathbb{C} \times \mathbb{C}^n$  and  $O(|\tilde{y}|^2)$  as  $\tilde{y} \rightarrow 0$ . Since (2.1) has the formal solution  $\hat{\tilde{y}}(x) = \sum_{m=M}^{\infty} c_{m+\mu} x^m$ , we see that  $\tilde{F}_0$  is analytic at 0.

Next we apply to the linear part of (2.1) the usual reduction procedure to a «normalized form» (cf. [9], [19], [11], [20]). Let  $k_1 > k_2 > \dots > k_r$  be the positive slopes of the Newton polygon



associated with  $x^{v+1} \frac{d}{dx} - \tilde{A}(x)$ . If  $M$  is sufficiently large then  $k_1, \dots, k_r$  are independent of  $M$ . We may assume that  $k_1, \dots, k_r$  are integers since otherwise we first perform a ramification  $x^* = x^{1/q}$  for some  $q \in \mathbb{N}$ . There exists an  $n \times n$ -matrix polynomial  $S(x)$  with  $\det S(x) \neq 0$  such that

$$S^{-1}(x) \left\{ x \frac{d}{dx} - \tilde{A}(x) \right\} S(x) = x \frac{d}{dx} - Q(x) - xA^*(x),$$

where  $A^*(x)$  is an analytic matrix in a neighborhood of 0 and

$$Q(x) = \bigoplus_{h=1}^{r+1} x^{-k_h} A_h.$$

Here  $A_h$  is an  $n_h \times n_h$ -matrix,  $n_h \in \mathbb{N}$ ,  $n_1 + \dots + n_{r+1} = n$ ,  $k_{r+1} = 0$  and if  $1 \leq h \leq r$  then  $A_h$  is invertible. Let  $\det S(x)$  have a zero of order  $\mu_0 \geq 0$  at  $x = 0$ . Then  $\mu_0$ ,  $A_1, \dots, A_r$  and  $A_{r+1} + \mu I_r$  are independent of  $M$  and  $\mu$  for sufficiently large  $M$ . Here  $I_h$  denote the  $n_h \times n_h$ -identity matrix.

Next we substitute  $\tilde{y}(x) = S(x)y^*(x)$  in (2.1). After deleting stars we obtain

$$x \frac{dy}{dx} - Q(x)y = xA(x)y + x^{M-v} S^{-1}(x) \tilde{F}_0(x) + x^{\mu-v} S^{-1}(x) \tilde{F}_2(x, S(x)y).$$

Because  $S^{-1}(x) = O(x^{-\mu_0})$  as  $x \rightarrow 0$  we choose  $\mu > v + \mu_0$  and  $M > v + \mu_0$  sufficiently large. Thus we see that (2.1) is equivalent with  $D[y] = 0$  where (cf. [16])

$$(2.2) \quad D[y](x) := \left( \bigoplus_{h=1}^{r+1} x^{k_h} I_h \right) x \frac{dy}{dx} - \left( \bigoplus_{h=1}^{r+1} A_h \right) y - G(x, y).$$

Here  $G(x, y)$  is analytic in  $\Delta_1(0; \rho) \times \Delta_n(0; \rho)$  for some  $\rho > 0$ ,

$$(2.3) \quad G(x, 0) = O(x^N) \text{ as } x \rightarrow 0, \quad G(x, y) = O(x) \text{ as } x \rightarrow 0,$$

uniformly in  $y$  on  $\Delta_n(0; \rho)$  with some  $N \in \mathbb{N}$ . Since  $D[y] = 0$  has a formal solution  $S^{-1}(x) \sum_{m=M}^{\infty} c_{m+\mu} x^m$  we see that we may choose  $N = M - \mu_0$  arbitrary large and that  $D[y] = 0$  has a formal solution  $\sum_{m=N}^{\infty} \tilde{c}_m x^m$ . Note that  $G$  depends on the choice of  $N$ .

We will consider the case  $r > 0$  only. If  $r = 0$  then  $D$  has a regular singularity in 0 and so the formal power series solution  $\hat{y}$  converges (theorem of Briot-Bouquet).

We now formulate the multisummability result for  $D[y] = 0$  in a somewhat more general case. Instead of  $G(x, y)$  analytic at  $(0, 0)$  in  $\mathbb{C} \times \mathbb{C}^n$  we assume that  $G(x, y)$  is the  $\mathbf{k}$ -sum in a multidirection  $\mathbf{d}$  of a formal power series  $\hat{G}(x, y)$  in  $x$ :

$$(2.4) \quad \hat{G}(x, y) := \sum_{m=1}^{\infty} G_m(y) x^m$$

with coefficients  $G_m(y)$  analytic on  $\Delta_n(0; \rho)$ . Here we use the notation of section 1 with  $\mathbf{k} = (k_1, \dots, k_r)$ , (1.10) and  $U_j := S(d_j, \varepsilon_j + \pi/k_j)$ ,  $U'_j := S(d_j, \varepsilon_j)$ . More precisely we assume:

a)  $U_{j-1} \subset U_j$ ,  $j = 2, \dots, r$ .

b)  $\hat{\mathcal{B}}_{k_r} \hat{G}(\cdot, y)(\xi)$  converges for  $0 < |\xi| < \rho_r$ , if  $y \in \Delta_n(0; \rho)$ , where  $\rho_r > 0$ ; its sum will be denoted by  $g_r(\xi, y)$ .

c) If  $j = r, \dots, 1$  respectively then  $g_j(\xi, y)$  can be analytically continued on  $U'_j \times \Delta_n(0; \rho)$  and  $g_j(\xi, y)$  is of exponential growth of order  $\mu_j$  at most as  $\xi \rightarrow \infty$  in  $U'_j$  uniformly for  $y \in \Delta_n(0; \rho)$ .

The latter means that to every closed subsector  $U'$  of  $U'_j$  correspond positive constants  $K$  and  $c_0$  such that

$$(2.5) \quad |g_j(\xi, y)| \leq K \exp |c_0 \xi^{\mu_j}| \quad \text{on } \{U' \setminus \Delta_1(0; 1)\} \times \Delta_n(0; \rho).$$

If  $j \neq 1$ , then

$$(2.6) \quad g_{j-1}(\xi, y) := \mathbf{A}_{k_{j-1}, k_j; d_j} g_j(\cdot, y)(\xi).$$

Moreover

$$(2.7) \quad g_j(\xi, y) \sim \hat{\mathcal{B}}_{k_j} \hat{G}(\cdot, y)(\xi) \quad \text{as } \xi \rightarrow 0 \\ \text{on } U'_j \text{ uniformly in } y \text{ on } \Delta_n(0; \rho).$$

d)  $G(x, y) := \{\mathcal{L}_{k_1, a_1} g_1(\cdot, y)\}(x)$  on  $U \times \Delta_n(0, \rho)$  where  $U$  is a neighborhood of 0 in  $U_1$ .

Then  $G(x, y)$  is  $\mathbf{k}$ -sum of  $\hat{G}(x, y)$  in  $\mathbf{d}$  and on  $\mathbf{U} = (U_1, \dots, U_r)$ . The assumptions above are satisfied if  $G(x, y)$  is analytic in  $(0, 0)$ : now  $\mathbf{d}$  and  $\mathbf{U}$  may be chosen arbitrarily.

To formulate the result we also use

**DEFINITION 2.** — Let  $j \in \{1, \dots, r\}$ . Then  $\xi$  is a singular value of level  $k_j$  for  $D$  and the direction of  $\xi$  is a singular direction of level  $k_j$  for  $D$  if  $k_j \xi^{k_j}$  is an eigenvalue of  $A$  (cf. (2.2)). The set of singular values and directions of level  $k_j$  for  $D$  will be denoted by  $V_j$  and  $W_j$  respectively. A direction  $\sigma$  is a Stokes direction for  $D$  if there exists  $d \in W_j$  for some  $j$  such that  $\arg \sigma = \arg d \pm \pi/(2k_j)$ .

Using the notation above we have

**THEOREM 1.** — Let  $D$  be given by (2.2) where

- (i)  $r \in \mathbb{N}$ ,  $k_{r+1} = 0 < k_r < \dots < k_1$ ,  $k_h \in \mathbb{N}$ ,  $I_h$  and  $A_h$  are  $(n_h \times n_h)$ -matrices,  $n_h \in \mathbb{N}$ ,  $n_1 + \dots + n_{r+1} = n$ ,  $I_h$  is identity matrix,  $h = 1, \dots, r+1$ , and if  $1 \leq h \leq r$  then  $A_h$  is invertible,
- (ii)  $G(x, y)$  is  $\mathbf{k}$ -sum in multidirection  $\mathbf{d}$  or multisector  $\mathbf{U}$  of  $\hat{G}(x, y)$  (cf. 2.4) satisfying a) - d) with  $\mathbf{k} = (k_1, \dots, k_r)$ ,  $\mathbf{d} = (d_1, \dots, d_r)$ ,  $\mathbf{U} = (U_1, \dots, U_r)$ ,  $U_j = S(d_j, \varepsilon_j + \pi/k_j)$ ,  $\varepsilon_j > 0$ ,  $j = 1, \dots, r$ .

Let  $\hat{y}(x) \in x\mathbb{C}[[x]]^n$  be a formal solution of  $D[y] = 0$ . Let  $\tau_j$  be a direction and  $\delta_j > 0$  such that  $S(\tau_j, \delta_j) \subset S(d_j, \varepsilon_j) \setminus W_j$  and  $S_j \subset S_{j+1}$  if  $j = 1, \dots, r$ , where  $S_j := S(\tau_j, \delta_j + \pi/k_j)$ .

Then  $\hat{y}(x)$  is  $\mathbf{k}$ -summable in multidirection  $\boldsymbol{\tau}$  and on multisector  $\mathbf{S}$  where  $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r)$ ,  $\mathbf{S} = (S_1, \dots, S_r)$ .

Its multisum  $y(x)$  is analytic solution of  $D[y] = 0$  in a neighborhood  $V$  of 0 in  $S_1$  and  $y(x) \sim \hat{y}(x)$  as  $x \rightarrow 0$  in  $V$ .

**Remark 1.** — If  $S_0$  is a subsector of  $U_1$  such that  $S_0$  does not contain a pair of Stokes directions  $\sigma \pm \pi/k_j$  with  $\sigma \in W_j$  then we may choose  $S_1, \dots, S_r$  in the theorem in such a way that the corresponding  $\mathbf{k}$ -sum  $y$  exists in a neighborhood  $V_0$  of 0 in  $S_0$ ,  $D[y] = 0$  on  $V_0$  and  $y(x) \sim \hat{y}(x)$  as  $x \rightarrow 0$  in  $V_0$ .

**COROLLARY.** — Combining theorem 1 with the reduction of (0.1) to  $D[y] = 0$  described in the beginning of this section we see that the formal power series solution  $\hat{y}$  in (0.2) of (0.1) is  $(k_1/q, \dots, k_r/q)$ -summable. Its multisums exist on all sectors which do not contain any pair of Stokes directions  $d_{\pm}$  with  $\arg d_{\pm} = q(\arg \sigma \pm \pi/(2k_j))$ ,  $\sigma \in W_j$ . The exceptional directions in which  $\hat{y}$  is not  $\mathbf{k}$ -summable are singular directions of (0.1).

The proof of Theorem 1 and Remark 1 will be given in sect. 3.

### 3. The convolution equations.

From the transformations in section 2 it follows that it is sufficient to consider the case that the first  $N$  terms in the formal solution vanish :

$$(3.1) \quad \hat{y}(x) = \sum_{m=N}^{\infty} c_m x^m$$

and that (2.3) holds with  $N$  sufficiently large.

First we consider  $D[y]$  with  $y = \mathcal{L}_{k_j} \psi$  for some  $j \in \{1, \dots, r\}$  where  $\psi \in C_0^\infty(d, \mathbb{C}^n)$  and  $d$  is a direction in  $U'_j$ . Then we apply the Borel transformation  $\mathcal{B}_{k_j}$  to a modification of  $D[\mathcal{L}_{k_j} \psi]$ . Here we utilize the relation

$$(3.2) \quad \left( \mathcal{B}_k x^{1+k} \frac{d}{dx} \mathcal{L}_k \psi \right) (\xi) = k \xi^k \psi(\xi).$$

Therefore we consider  $M_j D$  where

$$(3.3) \quad M_j = \bigoplus_{h=1}^{r+1} x^{m_{jh}} I_h \quad \text{with} \quad m_{jh} = \max(k_j - k_h, 0).$$

In this connection we use the following notation : if  $w \in \mathbb{C}^n$  then  $w^{(h)}$  denotes the projection of  $w$  onto the space spanned by the unit vectors of  $\mathbb{C}^n$  with indices  $n_1 + \dots + n_{h-1} + 1, \dots, n_1 + \dots + n_h$  where  $n_0 := 0$  and  $h \in \{1, \dots, r+1\}$ . Hence

$$(3.4) \quad \{M_j D[y]\}^{(h)} = x^{k_j+1} \frac{d}{dx} y^{(h)} - x^{k_j-k_h} A_h y^{(h)} - x^{k_j-k_h} G^{(h)}(x, y)$$

if  $h \geq j$  whereas if  $h < j$  we replace  $k_j$  by  $k_h$  in the right hand side.

Next we define an operator  $Q_j$  on  $C_0^\infty(d, \mathbb{C}^n)$  which formally is of the form  $\mathcal{B}_{k_j} M_j D \mathcal{L}_{k_j}$ . To give a meaning to the nonlinear part of this expression we use the function  $g_j(\xi, y)$  of condition c) in section 2, which corresponds to  $\hat{\mathcal{B}}_{k_j} \hat{G}(\cdot, y)$ . We have the expansion

$$(3.5) \quad g_j(\xi, y) = \sum_{m \in \mathbb{N}^n} g_{mj}(\xi) y^m, \quad \text{if } (\xi, y) \in U'_j \times \Delta_n(0; \rho).$$

Let  $\psi_{j*m} = \mathcal{B}_{k_j}(\mathcal{L}_{k_j}\psi)^m$  if  $m \in \mathbb{N}^n \setminus \{0\}$  and  $\psi$  as above. If  $m = m_1 e_1 + \dots + m_n e_n$  where  $e_h$  denotes the  $h^{\text{th}}$ -unit vector we have

$$(3.6) \quad \psi_{j*m} = \psi_{j*m_1 e_1} *_{k_j} \dots *_{k_j} \psi_{j*m_n e_n}, \quad \psi_{j*m_h e_h} = \psi^{(h)} *_{k_j} \dots *_{k_j} \psi^{(h)}$$

where in the last term  $(m_h - 1)$  convolutions  $*_{k_j}$  are performed.

If  $\psi \in C_0^\infty(d, \mathbb{C}^n)$  we define

$$(3.7) \quad g_{j*}(\xi, \psi) := \sum_{m \in \mathbb{N}^n} g_{mj}(\xi) *_{k_j} \psi_{j*m}.$$

In lemma 1 we will show that  $g_{j*}(\xi, \psi)$  makes sense if  $\psi \in C_0^\infty(d, \mathbb{C}^n)$ ,  $d \subset U'_j$  and also if  $\psi$  is analytic and bounded in a neighborhood of 0 in  $U'_j$ . In the latter case we define  $\psi_{j*m}$  by (3.6).

Let

$$(3.8) \quad D_0 := \bigoplus_{h=1}^{r+1} \left( x^{k_h+1} I_h \frac{d}{dx} - A_h \right), \quad \text{so } D_0[y] = D[y] + G(x, y).$$

We define the operator  $Q_j$  on  $C_0^\infty(d, \mathbb{C}^n)$  by:

$$(3.9) \quad Q_j \psi = \mathcal{B}_{k_j} M_j D_0 \mathcal{L}_{k_j} \psi - (\mathcal{B}_{k_j} M_j) *_{k_j} g_{j*}(\xi, \psi), \quad \psi \in C_0^\infty(d, \mathbb{C}^n).$$

From (3.2) and (3.4) we now derive, if  $k := k_j$ :

$$(3.10) \quad \begin{cases} (Q_j \psi)^{(h)} = -A_h \psi^{(h)} + \frac{\xi^{k_h-2k}}{\Gamma(-1+k_h/k)_k} * (k \xi^k \psi)^{(h)} - g_{j*}^{(h)}(\xi, \psi) & \text{if } h < j, \\ (Q_j \psi)^{(j)} = (k \xi^k I_j - A_j) \psi^{(j)} - g_{j*}^{(j)}(\xi, \psi), \\ (Q_j \psi)^{(h)} = k \xi^k \psi^{(h)} - \frac{1}{\Gamma(1-k_h/k)} \xi^{-k_h} * \{A_h \psi^{(h)} + g_{j*}^{(h)}(\xi, \psi)\} & \text{if } j < h \leq r+1. \end{cases}$$

Here the first terms in the righthand sides are the principal terms with the factors of  $\psi^{(h)}$  invertible if  $\xi \neq 0$  and  $\xi$  not a singular value of level  $k_j$ .

In view of lemma 1 we may extend the definition of  $Q_j \psi$  to analytic functions  $\psi$  in a neighborhood of 0 in  $U'_j$  with

$$(3.11) \quad \psi(\xi) \sim \sum_{l=N}^{\infty} \tilde{c}_l \xi^{l-k}, \quad \xi \rightarrow 0 \quad \text{in } U'_j.$$

We will solve  $Q_j\psi = 0$  with  $\psi(\xi) \sim \hat{\psi}_j(\xi)$  as  $\xi \rightarrow 0$  in a subsector of  $U'_j$ . Here

$$(3.12) \quad \hat{\psi}_j(\xi) := \hat{\mathcal{B}}_{k_j} \hat{y}(\xi) = \sum_{l=N}^{\infty} \frac{c_l}{\Gamma(l/k_j)} \xi^{l-k_j},$$

where  $N \geq k_1$  will be suitably chosen later on.

The analysis of the equations  $Q_j\psi = 0$  proceeds in 5 steps which we state as lemmas to be proved in later sections.

Let  $S'_j$  be a closed subsector of  $U'_j$  and  $U := S'_j \cap \overline{\Delta_1(0;p)}$  for some  $p > 0$ . Consider the space  $W(U)$  of continuous functions  $\psi: U \rightarrow \mathbb{C}^n$  such that

$$(3.13) \quad \|\psi\| := \sup_U |\xi^{k_j-N} \psi(\xi)| < \infty$$

and  $\psi$  is analytic in the interior of  $U$ . Then we have

LEMMA 1. — a) If  $\psi \in W(U)$  then

$$(3.14) \quad |\psi_{j\star m}(\xi)| \leq \frac{\{\Gamma(N/k_j)\|\psi\| |\xi^N|\}^{|m|}}{\Gamma(|m|N/k_j)} |\xi^{-k_j}| \quad \text{if } \xi \in U, \quad m \in \mathbb{N}^n \setminus \{0\}.$$

b) For  $g_{mj}$  defined by (3.5) we have :

$$(3.15) \quad |g_{mj}(\xi)| \leq K p^{-|m|} |\xi^{1-k_j}| \exp |c_0 \xi^{\mu_j}|, \quad \xi \in S'_j, \quad m \in \mathbb{N}^n \setminus \{0\}$$

$$(3.16) \quad |g_{oj}(\xi)| \leq K |\xi^{N-k_j}| \exp |c_0 \xi^{\mu_j}|, \quad \xi \in S'_j$$

where  $K$  and  $c_0$  are certain positive constants.

c) The series for  $g_{j\star}(\xi, \psi)$  in (3.7) is uniformly and absolutely convergent on  $U$  if  $\psi \in W(U)$ , in particular if  $\psi \in C_0^\infty(d, \mathbb{C}^n)$  and  $S'_j = d$ .

This lemma implies that  $Q_j$  is well defined on  $U'_j$ . The solution of  $Q_j\psi = 0$  will be deduced from the following lemmas :

LEMMA 2. — The equation  $Q_r\psi = 0$  has a unique solution  $\psi_r(\xi)$  which is the sum of the formal series  $\hat{\psi}_r(\xi)$  that converges on  $\Delta(0;r_1)$  for some  $r_1 > 0$ .

LEMMA 3. — Let  $j \in \{1, \dots, r\}$ . If  $j = r$  we denote by  $\psi_r$  the solution  $\psi_r$  in lemma 2. If  $j < r$  we assume that  $\psi_j$  is an analytic solution of  $Q_j\psi = 0$  on a neighborhood  $U$  of 0 in  $S(\tau_{j+1}, \delta_{j+1} + \pi/\mu_{j+1})$  and

$\psi_j(\xi) \sim \hat{\psi}_j(\xi)$  as  $\xi \rightarrow 0$  in  $U$ . Here  $\tau_{j+1}$  is some direction in  $U'_{j+1}$  and  $\delta_{j+1} > 0$ . Then  $\psi_j$  can be analytically continued on  $\tilde{S}_j$  and  $Q_j\psi_j = 0$  on  $\tilde{S}_j$  where  $\tilde{S}_r := U'_r \setminus V_r$  and  $\tilde{S}_j := S(\tau_{j+1}, \delta_{j+1} + \pi/\mu_{j+1}) \cap U'_j \setminus V_j$  if  $j < r$  with  $V_j$  given by definition 2.

LEMMA 4. — Let  $j$ ,  $\psi_j$  and  $\tilde{S}_j$  be as in lemma 3, and  $S(\tau_j, \delta_j) \subset \tilde{S}_j$  for some direction  $\tau_j$  and  $\delta_j > 0$ . Then  $\psi_j$  is of exponential growth of order  $\leq \mu_j$  in  $S(\tau_j, \delta_j)$ .

LEMMA 5. — Let  $j \in \{2, \dots, r\}$  and  $\psi_j$  be as in lemma 4. Then  $\psi_{j-1} := \mathbf{A}_{k_{j-1}, k_j} \psi_j$  is an analytic solution of  $Q_{j-1}\psi = 0$  in a neighborhood of 0 in  $S(\tau_j, \delta_j + \pi/\mu_j)$  and  $\psi_{j-1}(\xi) \sim \hat{\psi}_{j-1}(\xi)$  as  $\xi \rightarrow 0$  on this sector.

We postpone the proofs of these lemmas but now give the

*Proof of theorem 1.* — First apply lemmas 2, 3, 4 and 5 with  $j = r$  consecutively. Then we obtain an analytic solution  $\psi_{r-1}$  of  $Q_{r-1}\psi = 0$  in a neighborhood of 0 in  $S(\tau_r, \delta_r + \pi/\mu_r)$  with  $\psi_{r-1} \sim \hat{\psi}_{r-1}$ . Then we apply lemmas 3, 4 and 5 consecutively first for  $j = r - 1$ , then  $j = r - 2, \dots$ , until  $j = 2$ . Finally lemmas 3 and 4 with  $j = 1$  give an analytic solution  $\psi_1$  of  $Q_1\psi = 0$  on  $S(\tau_1, \delta_1)$  which is of exponential growth of order  $\leq \mu_1 = k_1$  (cf. (1.10)), and with  $\psi_1 \sim \hat{\psi}_1$ . Hence  $\hat{y}(x)$  is  $\mathbf{k}$ -summable in multidirection  $\tau$  and on multisector  $S$  with sum  $y(x) := (\mathcal{L}_{k_1, \tau_1} \psi_1)(x)$  on a neighborhood of 0 in  $S_1$ . Then it follows from (3.9) that

$$Q_1\psi_1 = \mathcal{B}_{k_1}\{M_1 D_0[y]\} - (\mathcal{B}_{k_1} M_1) *_{k_1} g_{1*}(\xi, \psi_1) = 0.$$

Hence using (3.7), (3.6), (3.15) and (3.5) we obtain

$$\begin{aligned} D_0[y] &= \mathcal{L}_{k_1} g_{1*}(\cdot, \psi_1) = \sum_{m \in \mathbb{N}^n} \mathcal{L}_{k_1} g_{m1}(\cdot) y^m = \mathcal{L}_{k_1} g_1(\cdot, y) \\ &= G(\cdot, y), \text{ i.e. } D[y] = 0 \text{ on } S_1. \quad \square \end{aligned}$$

*Proof of remark 1.* — The construction of  $S_1, \dots, S_r$  is similar to that in [5], end of sect. 2. Let  $S_0 = S(\tau_0, \delta_0 + \pi/k_1)$ ,  $\delta_0 > 0$ . Then there exists  $h \in \mathbb{N}$  such that  $\pi/k_{h-1} < \delta_0 + \pi/k_1 \leq \pi/k_h$ . If  $\tau_0$  is a singular direction and/or  $\delta_0 + \pi/k_1 = \pi/k_h$  we replace  $\tau_0$  by a nonsingular direction  $\tau_1$  near  $\tau_0$  and/or  $\delta_0$  by  $\delta_1$  near  $\delta_0$  such that  $\pi/k_{h-1} < \delta_1 + \pi/k_1 < \pi/k_h$  and  $S_1 := S(\tau_1, \delta_1 + \pi/k_1) \subset S_0$ . Then we choose  $S_j := S_1$  if  $j = 2, \dots, h - 1$ . Now  $S_j \subset U_1 \subset U_j$  and  $S_j$  does not contain a pair  $\sigma_j \pm \pi/(2k_j)$ ,  $\sigma_j \in W_j$ . Therefore  $S'_j := S(\tau_1, \delta_1 + \pi/k_1 - \pi/k_j)$  does not contain a singular direction of level  $k_j$  if  $j \leq h - 1$ .

Since the opening of  $S_1$  is less than  $\pi/k_j$  and  $S_1 \subset U_j \subset U_{j+1}$  if  $j \geq h$ , we may choose  $S_j$  for  $j \geq h$  in such a way that for  $j \geq h$  we have:  $S_1 \subset S_j \subset S_{j+1}$ ,  $S_j \subset U_j$ ,  $S_j$  does not contain a Stokes pair  $\sigma_j \pm \pi/(2k_j)$  with  $\sigma_j \in W_j$  and if  $S_j = S(\tau_j, \delta_j + \pi/k_j)$  then  $\delta_j > 0$ . Hence  $S(\tau_j, \delta_j) \cap W_j = \emptyset$ .

Thus  $\hat{y}$  is  $\mathbf{k}$ -summable on  $\mathbf{S}$  with multisum  $y$  on a neighborhood of 0 in  $S_1$ . If  $\tau_0 \neq \tau_1$  and/or  $\delta_0 \neq \delta_1$  we may vary  $\tau_1$  and/or  $\delta_1$  such that  $\tau_1 \rightarrow \tau_0$  and  $\delta_1 \rightarrow \delta_0$ . Then we obtain an extension of the multisum  $y$  on a neighborhood of 0 in  $S_0$  with  $y(x) \sim \hat{y}(x)$  as  $x \rightarrow 0$  in  $S_0$ .  $\square$

Finally we rewrite the equation  $Q_j\psi = 0$  (cf. (3.10)) in a form which is more suitable for the proofs of lemmas 2, 3 and 4. Let  $U$  be a star-region with vertex 0 in  $U'_j/V_j$ . Let  $W(U)$  be the space of analytic functions  $\psi: U \rightarrow \mathbb{C}^n$  such that  $\psi(\xi) = O(\xi^{1-k_j})$  as  $\xi \rightarrow 0$  on  $U$ . Then we define the operator  $T_j$  on  $W(U)$  as follows: let  $k := k_j$  and  $\psi \in W(U)$ ; then

$$(3.17) \quad \begin{cases} (T_j\psi)^{(h)} := A_h^{-1} \left\{ \frac{\xi^{k_h-2k}}{\Gamma(-1+k_h/k)_k} * (k\xi^k\psi)^{(h)} - g_{j*}^{(h)}(\xi, \psi) \right\} & \text{if } h < j, \\ (T_j\psi)^{(j)} := (k\xi^k I_j - A_j)^{-1} g_{j*}^{(j)}(\xi, \psi), \\ (T_j\psi)^{(h)} := \{\Gamma(1-k_h/k)k\xi^k\}^{-1} \{\xi^{-k_h} * (A_h\psi^{(h)} + g_{j*}^{(h)}(\xi, \psi))\} & \text{if } j < h \leq r+1. \end{cases}$$

Then  $Q_j\psi = 0$  is equivalent with  $T_j\psi = \psi$ .

#### 4. Proof of lemmas 1 and 2.

*Proof of lemma 1.* — Let  $k := k_j$ ,  $\mu := \mu_j$ . We have for  $\alpha > 0$ ,  $\beta > 0$ :

$$(4.1) \quad \xi^{\alpha-k} * \xi^{\beta-k} = B\left(\frac{\alpha}{k}, \frac{\beta}{k}\right) \xi^{\alpha+\beta-k},$$

where  $B$  denotes the beta-function. Hence

$$(4.2) \quad |(\xi^{\alpha-k} * \psi)(\xi)| \leq \|\psi\| |\xi^{\alpha-k} * \xi^{N-k}| = \|\psi\| B\left(\frac{\alpha}{k}, \frac{N}{k}\right) |\xi^{\alpha+N-k}|$$

if  $\xi \in U$ . Now a) follows by induction.



To prove b) we extend (2.5). From (2.3) and  $g_j(\xi, y) \sim \mathcal{B}_k G(\cdot, y)(\xi)$  as  $\xi \rightarrow 0$   $U'$  uniformly in  $\Delta_n(0; \rho)$  it follows that we may choose  $K$  in (2.5) so large that

$$|g_j(\xi, y)| \leq K |\xi^{1-k}| \exp |c\xi^\mu| \text{ on } U' \times \Delta_n(0; \rho),$$

and that (3.16) holds. From (3.5) and Cauchy's inequality we deduce (3.15). Assertion c) follows from (3.7), (3.15) and (3.14) with (4.1).

*Proof of lemma 2.* — The proof is analogous to that in [4], sect. 4.3. It is sufficient to prove theorem 1 for  $N$  sufficiently large in (3.1). We will choose  $N$  so large that  $T_r$  is a contraction.

Let  $k := k_r$ ,  $0 < p \leq p_0 < \text{dist}(V_r, 0)$  (cf. definition 2 for  $V_r$  in section 2). We define  $W(p)$  as the space of continuous functions  $\psi: \Delta_1(0, p) \rightarrow \mathbb{C}^n$  which are analytic in  $\Delta_1(0, p)$  and with

$$(4.3) \quad \|\psi\|_p := \frac{\max}{\Delta_1(0; p)} |\xi^{k-N} \psi(\xi)| < \infty.$$

First we estimate the linear part  $T_{lin}$  of  $T_r$  in  $W(p)$ . Recall  $k_{r+1} = 0$ . If  $\psi \in W(p)$  and  $\xi \in \Delta_1(0; p)$  we deduce from (4.1) in the same way as we obtained (4.2):

$$(4.4) \quad |\xi^{k_h-2k} *_k (\xi^k \psi)| \leq \|\psi\|_p B(-1 + k_h/k, 1 + N/k) |\xi^{N+k_h-k}| \text{ if } h < r,$$

$$|\xi^{-k} (1 *_k \psi)| \leq N^{-1} k \|\psi\|_p |\xi^{N-k}|.$$

Moreover, if  $|m| = 1$  we deduce from (3.15) and (4.1)

$$(4.5) \quad |g_{mr} *_k \psi_{r*m}| \leq \rho^{-1} K_1 \|\psi\|_p |\xi^{1-k} *_k \xi^{N-k}|$$

$$= \rho^{-1} K_1 \|\psi\|_p B(k^{-1}, Nk^{-1}) |\xi^{N+1-k}|$$

$$|\xi^{-1} (1 *_k g_{mr} *_k \psi_{r*m})| \leq \rho^{-1} (N+1)^{-1} k K_1 \|\psi\|_p B(k^{-1}, Nk^{-1}) |\xi^{N+1-k}|$$

where  $K_1 = K \exp(cp_0^\mu)$ . Because  $B(a, b) \rightarrow \infty$  as  $b \rightarrow \infty$  if  $a > 0$  we may deduce from (3.17), (3.7), (4.4), (4.5) and the definition of  $T_{lin}$  that there exists  $N_0 \in \mathbb{N}$ ,  $N_0 > k_1$  such that for all  $N \geq N_0$  and  $p \in (0, p_0]$  we have  $\|T_{lin}\|_p < \frac{1}{3}$  on  $W(p)$ . Therefore we choose  $N \geq N_0$  in (3.1).

Because of (3.17), (3.7) and (3.16) we have  $T_r(0) \in W(p_0) \subset W(p)$  if  $0 < p \leq p_0$ . Let  $R_0 := \|T_r(0)\|_{p_0}$ . Next we consider the higher order part of  $T_r$ : let  $\tilde{T} := T_r - T_{lin} - T_r(0)$ . We estimate  $\tilde{T}$  on  $B_p := \{\psi \in W(p) : \|\psi\|_p \leq 2R_0\}$  where  $0 < p \leq p_0$ .

The main ingredient in  $\tilde{T}\psi$  is

$$(4.6) \quad \tilde{g}(\xi, \psi) := \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geq 2}} g_{mr} * \psi_{r*m}.$$

Utilizing (3.15), (3.14) and (4.1) we derive that

$$(4.7) \quad |\tilde{g}(\xi, \psi)| \leq K' |\xi^{2N+1-k}| \quad \text{if } \psi \in B_p, \xi \in \overline{\Delta(0;p)}, 0 < p \leq p_0.$$

Here  $K'$  is a positive constant independent of  $\psi$ ,  $\xi$  and  $p$ . Hence  $\tilde{g}(\xi, \psi) \in B_p$  if  $p$  is sufficiently small. Next we consider  $\tilde{g}(\xi, \psi + \chi) - \tilde{g}(\xi, \psi)$  with  $\psi$  and  $\psi + \chi \in B_p$ . From (3.14) and (4.1) we deduce that on  $\overline{\Delta_1(0;p)}$  we have

$$\begin{aligned} |(\psi + \chi)_{r*m} - \psi_{r*m}| &= \left| \sum_{\substack{l \in \mathbb{N}^n \\ 0 \neq l \leq m}} \binom{m}{l} \psi_{r*(m-l)} * \chi_{r*l} \right| \\ &\leq \{\Gamma(|m|N/k)\}^{-1} \{\Gamma(N/k) |\xi^N|\}^{|m|} |\xi^{-k}| \sum_{\substack{l \in \mathbb{N}^n \\ 0 \neq l \leq m}} \binom{m}{l} \|\psi\|_p^{m-l} \|\chi\|_p^{|l|} \\ &\leq \{2R_0\Gamma(|m|N/k)\}^{-1} \{4R_0\Gamma(N/k) |\xi^N|\}^{|m|} |\xi^{-k}| \|\chi\|_p \quad \text{if } \xi \in \overline{\Delta_1(0;p)}. \end{aligned}$$

From this, (4.6), (3.15) and (4.1) we derive that

$$|\tilde{g}(\xi, \psi + \chi) - \tilde{g}(\xi, \psi)| \leq K'' \|\chi\|_p |\xi^{2N+1-k}|,$$

if  $\psi, \psi + \chi \in B_p$ ,  $\xi \in \overline{\Delta_1(0;p)}$ , where  $K''$  is a positive constant independent of  $\psi$ ,  $\chi$ ,  $\xi$  and  $p$  if  $0 < p \leq p_0$ .

From this, (4.7), (3.17), the definition of  $\tilde{T}$  and (4.4) it follows by a similar reasoning as above for  $T_{lin}$  that there exists a positive constant  $K'''$  independent of  $\psi$ ,  $\chi$ ,  $\xi$  and  $p$  such that

$$|(\tilde{T}\psi)(\xi)| \leq K''' |\xi^{2N+1-k}|, \quad |\{\tilde{T}(\psi + \chi) - \tilde{T}\psi\}(\xi)| \leq K''' \|\chi\|_p |\xi^{2N+1-k}|$$

with  $\psi$ ,  $\chi$ ,  $\xi$  and  $p$  as above. Hence there exists  $p \in (0, p_0]$  such that

$$\|\tilde{T}\psi\|_p \leq \frac{1}{3} R_0, \quad \|\tilde{T}(\psi + \chi) - \tilde{T}\psi\| \leq \frac{1}{3} \|\chi\|_p \quad \text{if } \psi, \psi + \chi \in B_p.$$

Combining this with the estimates in  $T_{lin}$  and  $T_r(0)$  we see that  $T_r$  is a contraction on  $B_p$ . Therefore we get a unique solution  $\psi_r$  in  $B_p$ .

For every  $M \geq N_0$  we obtain a unique solution  $\psi_{r,M}$  depending on  $M$  on a disc around 0. From the construction of  $Q_r$  and  $T_r$  from  $D$  and (0.2) it follows that

$$\psi_{r,N}(\xi) - \psi_{r,M}(\xi) = B_{k_r} \left( \sum_{m=N}^{M-1} c_m x^m \right) (\xi) \text{ if } M > N \geq N_0.$$

Hence if  $\hat{y}(x)$  is given by (3.1) and  $\psi_r$  corresponds with  $\psi_{r,N}$  then the Taylor coefficients of  $\psi_r$  and  $\mathcal{B}_{k_r} \hat{y}(\xi) = \hat{\psi}_r(\xi)$  are the same. So  $\hat{\psi}_r$  converges and  $\psi_r(\xi)$  is the sum of  $\hat{\psi}_r(\xi)$  on  $\Delta_1(0;p)$ .

### 5. Analytic continuation of solutions.

Let  $S'(\rho') := \{\xi \in \mathbb{C} \mid \alpha \leq \arg \xi \leq \beta, 0 < |\xi| \leq \rho'\}$  be a bounded closed subsector of  $U'_j \setminus V_j$  on which a continuous solution  $\psi_j$  of  $T_j \psi = \psi$  exists such that  $\psi_j$  is analytic in the interior of  $S'(\rho')$  and  $\psi_j(\xi) \sim \hat{\psi}_j(\xi)$  as  $\xi \rightarrow 0$  on  $S'(\rho')$ . ( $V_j$  is defined as set of singular values of level  $k_j$  in section 2.) We prove lemma 3 in this section and show that  $\psi_j$  can be continued analytically in  $U'_j \setminus V_j$  by linearizing  $Q_j \psi = 0$  in a subsector (cf. [6], [4], section 4.3). We fix  $j$  and denote  $k := k_j$ .

Choose  $\xi_0 \in S'(\rho')$  with  $|\xi_0| =: \rho_1$ ,  $0 < \rho_1 < \rho'$ . Let  $p \in (0, \rho_1]$ ,  $S''$  a subsector of  $S'$  with vertex 0,  $S''(p) = \{\xi \in S'' \mid 0 < |\xi| \leq p\}$  such that  $\tilde{S} := \{\xi \in \mathbb{C} \mid (\xi^k - \xi_0^k)^{1/k} \in S''(p) \text{ or } \xi = \xi_0\}$  satisfies  $\tilde{S} \subset U'_j \setminus V_j$  and  $\tilde{S} \cap S'(\rho_1) = \{\xi_0\}$ .

For the analytic continuation of  $\psi_j$  on  $\tilde{S}$  we utilize the space  $W$  of continuous functions  $\varphi: \tilde{S} \rightarrow \mathbb{C}^n$  which are analytic in the interior of  $\tilde{S}$ . Let  $S_0 := S'(\rho_1) \cup \tilde{S}$  and  $W_0$  be the space of functions  $\varphi: S_0 \rightarrow \mathbb{C}^n$  which are continuous on  $S_0 \setminus \{\xi_0\}$  and analytic in its interior whereas  $\lim \varphi(\xi)$  exists as  $\xi \rightarrow \xi_0$  on  $|\xi| < \rho_1$  and  $|\xi| > \rho_1$  respectively.

If  $\varphi \in W$  we define  $\varphi_0 = \varphi$  on  $\tilde{S}$  and  $\varphi_0 = 0$  on  $S'(\rho_1) \setminus \{\xi_0\}$ . Then  $\varphi_0 \in W_0$ . Moreover we define  $\tilde{\psi} \in W_0$  by  $\tilde{\psi} = \psi_j$  on  $S'(\rho_1)$  and  $\tilde{\psi} = \psi_j(\xi_0)$  on  $\tilde{S}$ .

If  $\varphi, \chi \in W_0$  we define  $\varphi \tilde{*} \chi$  as follows:  $\varphi \tilde{*} \chi = \varphi * \chi$  on  $S'(\rho_1)$  whereas if  $\xi \in \tilde{S} \setminus \{\xi_0\}$  we define

$$(\varphi \tilde{*} \chi)(\xi) = \int_{C(\xi)} \varphi((\xi^k - t^k)^{1/k}) \chi(t) d(t^k)$$

where  $C(\xi)$  is a path from 0 to  $\xi$  in  $S_0$  (so  $\xi_0 \in C(\xi)$ ) such that  $t \in C(\xi)$  implies  $(\xi^k - t^k)^{1/k} \in S_0$ . In particular if  $t \in C(\xi) \cap \tilde{S}$  then  $(\xi^k - t^k)^{1/k} \in S'(\rho_1)$ . Such paths  $C(\xi)$  exist because of the definition of  $\tilde{S}$  and  $S_0$ . Now  $\varphi \tilde{*} \chi \in W_0$ .

If the  $m$ -fold convolution of  $\chi \in W_0$  with respect to  $\tilde{*}$  is denoted by  $\chi_{\tilde{*}m}(m \in \mathbb{N}^n)$ , then we have  $\tilde{\Psi}_{\tilde{*}m} = (\psi_j)_{j \tilde{*} m}$  on  $S'(\rho_1)$  and if  $\varphi \in W$  and  $|m| \geq 2$  then  $(\varphi_0)_{\tilde{*}m} = 0$  on  $S_0$ .

From this we may deduce

$$(\tilde{\Psi} + \varphi_0)_{\tilde{*}m} = \tilde{\Psi}_{\tilde{*}m} + \sum_{l=1}^n m_l \tilde{\Psi}_{\tilde{*}(m-e_l)} \tilde{*} \varphi_0^{(l)},$$

where  $m = (m_1, \dots, m_n)$  and  $e_l$  denotes the  $l$ -th unit vector. Let

$$\tilde{g}(\xi, \chi) := g_{0j}(\xi) + g_{1j}(\xi) \tilde{*} \chi(\xi) + \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geq 2}} g_{mj}(\xi) \tilde{*} \chi_{\tilde{*}m}(\xi) \text{ if } \chi \in W_0.$$

Then it follows that for  $\varphi \in W$  we have

$$\tilde{g}(\xi, \tilde{\Psi} + \varphi_0) = \tilde{g}(\xi, \tilde{\Psi}) + B(\xi) \tilde{*} \varphi_0(\xi)$$

where  $B(\xi)$  is an  $(n \times n)$ -matrix valued function with  $(h, l)$ -element  $B^{(h, l)}(\xi)$  given by

$$B^{(h, l)}(\xi) = g_{1j}^{(h, l)}(\xi) + \sum_{\substack{m \in \mathbb{N}^n \\ |m| \geq 2}} g_{mj}^{(h)}(\xi) \tilde{*} m_l \tilde{\Psi}_{\tilde{*}(m-e_l)}(\xi).$$

To get a solution of  $\psi = T_j \psi$  on  $\tilde{S}$  we substitute  $\psi = \tilde{\Psi} + \varphi_0$  with  $\varphi \in W$  in this equation, and replace the convolutions  $*$  in the definition of  $T_j$  by  $\tilde{*}$ . Then the equation for  $\varphi$  becomes  $\varphi = L\varphi + \chi$  where  $\chi = \{(T_j \tilde{\Psi}) - \tilde{\Psi}\}|_{\tilde{S}}$  and  $L$  is the linear operator given by

$$\begin{aligned} (L\varphi)^{(h)} &:= A_h^{-1} [\{\Gamma(-1 + k_h/k)\}^{-1} \xi^k h^{-2k} \tilde{*} (k \xi^k \varphi_0) - B(\xi) \tilde{*} \varphi_0]^{(h)} \text{ if } h < j, \\ (L\varphi)^{(j)} &= (k \xi^k I_j - A_j)^{-1} B(\xi) \tilde{*} \varphi_0^{(j)}, \\ (L\varphi)^{(h)} &:= \{\Gamma(1 - k_h/k) k \xi^k\}^{-1} [\xi^{-k_h} \tilde{*} \{A_h \varphi_0 + B(\xi) \tilde{*} \varphi_0\}^{(h)}] \text{ if } h > j, \end{aligned}$$

where in the right hand sides we take the restrictions to  $\tilde{S}$ .

Let  $\|\varphi\|$  be the supremum norm of  $\varphi \in W$ . Then for  $\xi \in \tilde{S}$  we have

$$|B(\xi) \tilde{*} \varphi_0(\xi)| = \left| \int_{\xi_0}^{\xi} B((\xi^k - t^k)^{1/k}) \varphi(t) d(t^k) \right| \\ \leq \|\varphi\| \int_0^{\eta} |B(\tau) d(\tau^k)| \text{ where } \eta = (\xi^k - \xi_0^k)^{1/k} \in S''(p) \subset S'(\rho').$$

Similarly for  $B$  replaced by  $\xi^{-k_h} \tilde{*} B$ . Hence  $L$  only depends on the values of  $B$  in  $S''(p)$ , so on  $\psi_j$  in  $S''(p)$ .

Therefore there exists  $p_0 > 0$  independent of  $\rho'$  and  $\rho_1$  such that for  $0 < p \leq p_0$  we have  $\|L\| < 1$ . Hence we get a unique solution  $\varphi \in W$  of  $\varphi = L\varphi + \chi$  if  $p \leq p_0$ . Thus  $\psi = \varphi + \psi_j(\xi_0)$  is unique solution of  $T_j\psi = \psi$  on  $\tilde{S}$ . If  $p < \rho' - \rho_1$  then we have already the analytic solution  $\psi_j$  on  $\tilde{S} \subset S'(\rho')$ . Hence  $\varphi + \psi_j(\xi_0)$  is analytic continuation of  $\psi_j$  on  $\tilde{S}$  if  $\rho' - \rho_1 = \rho' - |\xi_0| < p < p_0$ . Repeated application of this procedure gives the analytic continuation of  $\psi_j$  on the star region with vertex 0 in  $U'_j \setminus V_j$ . Moreover, we obtain analytic continuation on the singular directions in  $U'_j$  outside the singular points of level  $k_j$ . This implies lemma 3. We remark that the analytic continuation may be obtained arbitrary on Riemann surfaces above  $U'_j \setminus V_j$  by means of more complicated symmetric paths of integration for the convolution integrals as in Ecalle [6].

## 6. Exponential estimates.

In this section we prove lemma 4 using a majorant equation for  $\psi = T_j\psi$  and the following lemma.

LEMMA 6. — *Let  $a > 0$ ,  $b \geq 0$ ,  $\mu > 1$  and  $a + b \leq \mu a$ . Then there exists a constant  $K > 0$  such that for all  $c \geq 1$  we have for all positive  $p$*

$$(6.1) \quad p^{a-1} * \{p^b \exp(cp^\mu)\} \leq Kc^{-(a+b)/\mu} \exp(cp^\mu)$$

and if  $a \leq \mu$

$$(6.2) \quad \{p^{a-1} \exp(cp^\mu)\} * \exp(cp^\mu) \leq Kc^{-a/\mu} \exp(cp^\mu).$$

We postpone the proof of lemma 6 till the end of this section. Let  $S'$  be a closed subsector of  $\tilde{S}_j$  (cf. lemma 3) with vertex 0 and let

$k := k_j$ ,  $\mu := \mu_j/k_j$ ,  $p_1 > 0$  and  $\psi_j$  be the analytic solution of  $Q_j\psi = 0$  as in lemma 3. We define  $\Psi(p) := \sup \{ |\psi_j(\xi)| : \xi \in S', |\xi| = p^k \}$  if  $p > 0$ . We first derive an integral inequality for  $\Psi$ . From (3.15) we deduce that if  $|\xi| = p^k$  and  $j < h \leq r+1$  then

$$\begin{aligned} |\xi^{-k_h} * g_{mj}(\xi)| &\leq K \rho^{-|m|} \{ p^{-k_h/k} * (p^{-1+1/k} \exp(c_0 p^\mu)) \} \\ &\leq K \rho^{-|m|} \{ \exp(c_0 p^\mu) \} B(1 - k_h/k, 1/k) p^{(1-k_h)/k} \quad \text{if } |m| \geq 1. \end{aligned}$$

A similar estimate holds for  $m = 0$  (cf. (3.16)).

From this, (3.17),  $T_j\psi_j = \psi_j$  and (3.7) it follows that there exist positive constants  $l$  and  $q$  such that for  $p \geq 1$  we have  $\Psi(p) < (M\Psi)(p)$  where

$$\begin{aligned} (M\Psi)(p) = l \left[ \exp(2c_0 p^\mu) + \sum_{h=1}^{j-1} p^{-2+k_h/k} * (p\Psi) + \sum_{h=j+1}^{r+1} p^{-k_h/k} * \Psi \right. \\ \left. + \left\{ \sum_{h=j}^{r+1} p^{(1-k_h)/k} \exp(c_0 p^\mu) \right\} * \sum_{m=1}^{\infty} q^m \Psi_{*m} \right](p), \end{aligned}$$

with  $\Psi_{*m}$  the ordinary  $m$ -fold convolution of  $\Psi$ . Since  $\Psi(p)$  is bounded on  $(0, 1)$  we may choose  $l$  so large that  $\Psi(p) < (M\Psi)(p)$  also holds for  $0 < p < 1$ .

We now consider separately the cases  $1 < j \leq r$  and  $j = 1$ .

I. Case  $1 < j \leq r$ . Then  $\mu > 1$ . Let  $v(p) := l \exp(cp^\mu)$  where  $c > 2c_0$ ,  $c > 1$ . We show that it is possible to choose  $c$  so large that  $v(p) \geq (Mv)(p)$ . From lemma 6 it follows that  $v_{*m}(p) \leq (lKc^{-1/\mu})^{m-1} v(p)$ . Next we apply lemma 6 to the other terms in  $Mv$ . Here we use also that if  $\alpha > \mu$  and  $c \geq 2c_0$  then there exists a positive  $K_0$  independent of  $c$  such that  $p^{\alpha-1} \exp(c_0 p^\mu) \leq K_0 p^{\mu-1} \exp(cp^\mu)$ .

Thus we may derive

$$(Mv)(p) \leq l[\exp(2c_0 p^\mu) + K_2 c^{-1/\mu} v(p)]$$

if  $qlKc^{-1/\mu} < \frac{1}{2}$ . Here  $K_2$  is a constant independent of  $c$ . Thus we see

that we may choose  $c$  so large that  $Mv \leq v$ . Now  $\Psi < M\Psi$ ,  $\Psi(0) < v(0)$ ,  $Mv \leq v$  and  $M$  is a monotone operator. So if  $\Psi(t) < v(t)$  on  $0 \leq t < p$  then  $\Psi(p) < M\Psi(p) < Mv(p) \leq v(p)$ . Hence  $\Psi(p) < v(p)$ ,  $\forall p > 0$  and so  $\psi_j$  is of exponential growth of order  $\leq \mu_j$  if  $j < r$ .

II. Case  $j = 1$ . We now have  $\mu = 1$  and lemma 6 does not apply. We proceed similarly as in [4], section 4.5 and solve  $v = Mv$  using Laplace transforms. Let  $u := L_1 v$ . Then  $v = Mv$  becomes

$$u(x) = l \left[ (x^{-1} - 2c_0)^{-1} + \sum_{h=2}^{r+1} \Gamma(1 - k_h/k) x^{1-k_h/k} u(x) \right. \\ \left. + \sum_{h=1}^{r+1} \Gamma(1 + (1 - k_h)/k) (x^{-1} - c_0)^{-1 - (1 - k_h)/k} (1 - qu(x))^{-1} qu(x) \right].$$

This equation has a unique solution  $u$  which is analytic in  $x^{1/k}$  in a neighborhood of 0, real valued for  $x > 0$  and  $u(x) = lx(1 + o(1))$  as  $x \rightarrow 0$ . Hence we obtain a solution  $v = \mathcal{B}_1 u$  of  $v = Mv$ . Then  $v(p)$  is real-valued for  $p > 0$ ,  $v(p) = O(\exp(cp))$  as  $p \rightarrow \infty$  we choose  $c > 2c_0$ ,  $v(p) = l(1 + o(1))$  as  $p \rightarrow 0$ . Again monotonicity of  $M$  now implies  $\Psi(p) < v(p)$  and so  $\psi_1$  is of exponential growth of order  $\leq k_1$ .

*Proof of lemma 6.* — A proof of (6.1) has been given in [5], (I) in section 3. To prove (6.2) we use

$$(6.3) \quad \{p^{a-1} \exp(cp^\mu)\} * \exp(cp^\mu) = p^a \int_0^1 g(t) dt$$

where

$$g(t) := t^{a-1} \exp[cp^\mu\{t^\mu + (1-t)^\mu\}].$$

We now use that  $f(t) := t^\mu + (1-t)^\mu$  is convex on  $0 \leq t \leq 1$  since  $\mu > 1$ . Hence if  $0 \leq t \leq \frac{1}{2}$  then  $f(t) \leq (1-2t)f(0) + 2tf(\frac{1}{2}) = 1 - \mu_1 t$  where  $\mu_1 = 2(1 - 2^{1-\mu}) > 0$ . Therefore

$$(6.4) \quad \int_0^{\frac{1}{2}} g(t) dt \leq \int_0^{\frac{1}{2}} t^{a-1} \exp\{cp^\mu(1 - \mu_1 t)\} dt \\ \leq \Gamma(a) \{c\mu_1 p^\mu\}^{-a} \exp(cp^\mu).$$

Similarly

$$(6.5) \quad \int_{\frac{1}{2}}^1 g(t) dt \leq (1 + 2^{1-a}) \int_0^{\frac{1}{2}} \exp\{cp^\mu(1 - \mu_1 t)\} dt \\ \leq (1 + 2^{1-a})(c\mu_1 p^\mu)^{-1} \exp(cp^\mu).$$

If  $cp^\mu \geq 1$  then

$$p^a (cp^\mu)^{-a} = c^{-a/\mu} (cp^\mu)^{-a+a/\mu} \leq c^{-a/\mu}$$

and

$$p^a (cp^\mu)^{-1} = c^{-a/\mu} (cp^\mu)^{-1+a/\mu} \leq c^{-a/\mu}.$$

If  $cp^\mu \geq 1$  we now obtain (6.2) by combining these estimates with (6.3), (6.4) and (6.5). If  $cp^\mu \leq 1$  then  $p^a \leq c^{-a/\mu}$  and  $\int_0^1 g(t) dt \leq \int_0^1 t^{a-1} e dt = e/a$  and now also (6.2) follows.

## 7. Proof of lemma 5.

The proof of lemma 5 is analogous to that of lemma 3 in [5]. Let  $\psi \in C_0^\infty(d, \mathbb{C}^n)$  where  $d$  is a direction in  $S(\tau_j, \delta_j)$  and

$$(7.1) \quad \tilde{\psi} = A_{k_{j-1}, k_j} \psi.$$

From the definition of  $Q_j$  in (3.9) we deduce

$$\begin{aligned} Q_{j-1} \tilde{\psi} &= \mathcal{B}_{k_{j-1}}(M_{j-1} D_0 \mathcal{L}_{k_j} \psi) - (\mathcal{B}_{k_{j-1}} M_{j-1}) *_{k_{j-1}} g_{j-1*}(\xi, \tilde{\psi}) \\ &= \mathcal{B}_{k_{j-1}}(M_{j-1} M_j^{-1}) *_{k_{j-1}} \chi \end{aligned}$$

where

$$\chi = \mathcal{B}_{k_{j-1}}(M_j D_0 \mathcal{L}_{k_j} \psi) - (\mathcal{B}_{k_{j-1}} M_j) *_{k_{j-1}} g_{j-1*}(\xi, \tilde{\psi}).$$

From (3.7), (1.9) and lemma 1 we may derive

$$g_{j-1*}(\xi, \tilde{\psi}) = A_{k_{j-1}, k_j} g_{j*}(\xi, \psi).$$

Hence

$$\chi = A_{k_{j-1}, k_j} \{ \mathcal{B}_{k_j}(M_j D_0 \mathcal{L}_{k_j} \psi) - (\mathcal{B}_{k_j} M_j) *_{k_j} g_{j*}(\xi, \psi) \} = A_{k_{j-1}, k_j} (Q_j \psi),$$

and so

$$(7.2) \quad Q_{j-1} \tilde{\psi} = \mathcal{B}_{k_{j-1}}(M_{j-1} M_j^{-1}) *_{k_{j-1}} A_{k_{j-1}, k_j} (Q_j \psi).$$

By a density argument we may extend (7.2) with (7.1) to the case that  $\Psi \in C^\infty(d, \mathbb{C}^n)$ ,  $\psi$  continuous at 0 on  $d$  and of exponential growth of order  $\leq \mu_j$  at  $\infty$  on  $d$ . So (7.2) holds for  $\psi_j$  as in lemma 4. Hence  $Q_{j-1} \psi_{j-1} = 0$  on the set where  $A_{k_{j-1}, k_j} \psi_j$  exists, so on a neighborhood of 0 in  $S(\tau_j, \delta_j + \pi/\mu_j)$ . Furthermore  $\psi_{j-1} \sim \hat{\psi}_{j-1}$  at 0 in this sector because of (1.8) and (3.11).



### 8. Stokes phenomenon.

Suppose  $\tau_0$  is a singular direction for  $D[y]$  and  $k$  is the highest level for which  $\tau_0$  is singular (cf. section 2). Assume moreover that  $G(\cdot, y)$  is multisum of  $\hat{G}(\cdot, y)$  in all directions  $\tau$  in a neighborhood of  $\tau_0$ . Then theorem 1 gives two solutions  $y_+$  and  $y_-$  of  $D[y] = 0$  which are multisums of  $\hat{y}(x)$  in all directions  $\tau$  with  $\arg \tau_0 < \arg \tau < \arg \tau_0 + \varepsilon$  and  $\arg \tau_0 - \varepsilon < \arg \tau < \arg \tau_0$  respectively. Hence  $y_{\pm}(x) \sim \hat{y}(x)$  and  $y_+(x) - y_-(x) \sim 0$  as  $x \rightarrow 0$  on  $S(\tau_0, \pi/k_1)$ . Therefore  $y_+$  and  $y_-$  exhibit a Stokes phenomenon.

From the construction of the multisums  $y_+$  and  $y_-$  given in section 3 we may extract more precise information on  $y_+ - y_-$  in an analogous way as in [5]. In particular we may show that

$$y_+(x) - y_-(x) = O(1) \exp(-(c/x)^k) \text{ as } x \rightarrow 0 \quad \text{on} \quad S(\tau_0, \pi/k)$$

where  $c \in (0, c_0)$  arbitrary close to  $c_0$ , if  $c_0$  is the singular value of level  $k$  which is closest to the origin. It is possible to give more accurate estimates for  $y_+ - y_-$  using the analysis of [5]. Compare also Sibuya [17], [18] and Ramis and Sibuya [16].

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